

5.3 Confluence Without Termination

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In prog. languages: evaluation strategy may influence efficiency, but it should not influence the result.

But programs do not always terminate.

⇒ Programs should impose syntactic restrictions that ensure confluence even in case of non-termination.

(Indeed, languages like Haskell impose such restrictions.)

Requiring that all critical pairs are joinable is sufficient for local confluence, but not for confluence if the TRS is not terminating.

First Idea: Require that there must not be any critical pairs.

Def 5.3.1. (Non-Overlapping TRSs)

A TRS is non-overlapping iff it has no critical pairs.

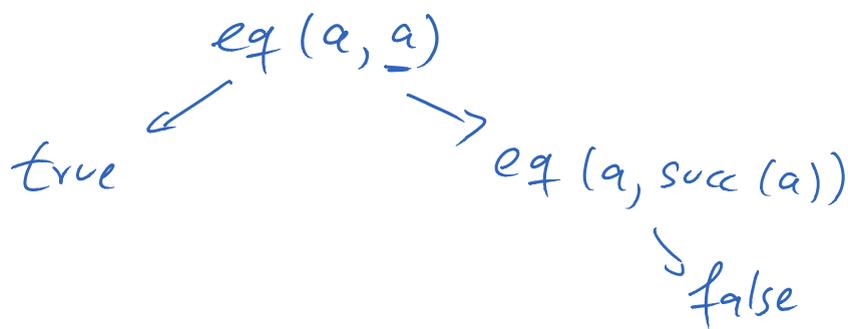
Unfortunately, this is not enough to ensure confluence.

Ex. 532 $eq(x, x) \rightarrow true$

$eq(x, succ(x)) \rightarrow false$

$a \rightarrow succ(a)$

This TRS is non-overlapping, but not confluent:



The problem is that the left-hand sides of the first 2 rules are not linear.

Def 533 (Left-linearity, Orthogonality)

A term is linear iff it does not contain multiple occurrences of the same variable. A TNS is left-linear iff all left-hand sides of its rules are linear. A TNS is orthogonal iff it is non-overlapping and left-linear.

We will show that: orthogonal \leadsto confluent
(even for non-terminating TNSs).

This is the reason why left-linearity is required in functional prog. languages.

To show this: introduce "strong confluence".

Def 534 (Strong Confluence)

A relation \rightarrow on a set M is strongly confluent iff for all $p, s, t \in M$:

if $p \rightarrow s$ and $p \rightarrow t$,

then there exists a $q \in M$ such that

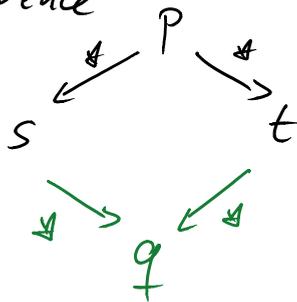
$$s \rightarrow^= q \text{ and } t \rightarrow^= q.$$

Here, " $\rightarrow^=$ " is the reflexive closure of " \rightarrow ";

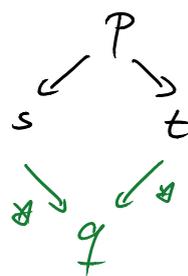
i.e., $s \rightarrow^= q$ iff $s \rightarrow q$ or $s = q$.

A TRS R is strongly confluent iff \rightarrow_R is strongly confluent.

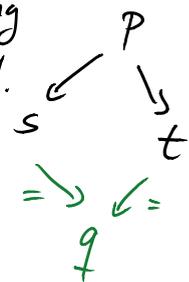
Confluence



Local Confluence



Strong Confl.



Ex 5.3.5 $R = \{ b \rightarrow a, b \rightarrow c, a \rightarrow b, c \rightarrow b \}$

$$a \hookrightarrow b \rightrightarrows c$$

R is strongly confluent.

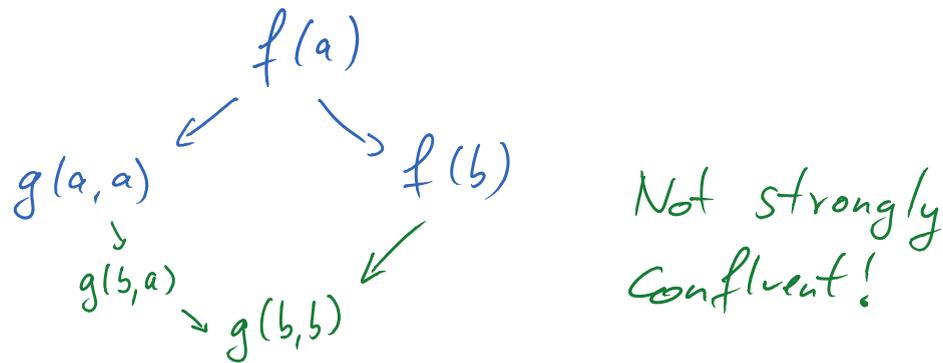
Thm 5.3.6 (Strong Confl. \Rightarrow Confl.)

Let \rightarrow be a strongly confluent relation. Then \rightarrow is confluent.

—
Is \rightarrow_R strongly confluent for every orthogonal TRS R ? If yes, then this would prove that every orthogonal TRS is confluent.

Ex: $R = \{ f(x) \rightarrow g(x,x), a \rightarrow b \}$

\mathcal{R} is orthogonal. (and confluent, since it is terminating and has no critical pairs.)



Idea: introduce a variant of the rewrite relation $\rightarrow_{\mathcal{R}}$ which allows to reduce subterms on pairwise independent positions in parallel. This variant is indeed strongly confluent for orthogonal TNSs \mathcal{R} .

Def 5.3.7 (Parallel Rewrite Relation)

For a TNS \mathcal{R} , we define the parallel rewrite relation $\Rightarrow_{\mathcal{R}}$ as follows: Let $s \Rightarrow_{\mathcal{R}} t$ hold iff

there is a set $\Pi = \{\pi_1, \dots, \pi_n\} \subseteq O_{cc}(s)$ with $n \neq 0$

and for each π_i there is a rule $l_i \rightarrow r_i \in \mathcal{R}$ and a

substitution σ_i such that

$$s|_{\pi_i} = l_i \sigma_i \text{ and } t = s[r_1 \sigma_1]_{\pi_1} \dots [r_n \sigma_n]_{\pi_n}.$$

Here, $\pi_i \perp \pi_j$ must hold for all $1 \leq i < j \leq n$.

(All elements of Π must be pairwise independent.)

Ex. 5.3.8. $\mathcal{R} = \{f(x) \rightarrow g(x, x), a \rightarrow b\}$

$g(a, a)$ has the positions $\Pi = \{1, 2\}$.

Thus $g(a, a) \Rightarrow_{\mathcal{R}} g(b, b)$.

To prove that orthogonality implies confluence, we show:

\mathcal{R} orthogonal $\overset{\text{TODO}}{\rightsquigarrow} \Rightarrow_{\mathcal{R}}$ is strongly confluent

$\rightsquigarrow \Rightarrow_{\mathcal{R}}$ is confluent
Thm 5.3.6

$\overset{\text{TODO}}{\rightsquigarrow} \rightarrow_{\mathcal{R}}$ is confluent

Lemma 5.3.9 (Confluence of $\rightarrow_{\mathcal{R}}$ and $\Rightarrow_{\mathcal{R}}$)

Let \mathcal{R} be a TRS.

(a) $\rightarrow_{\mathcal{R}} \subseteq \Rightarrow_{\mathcal{R}} \subseteq \rightarrow_{\mathcal{R}}^*$

(b) $\rightarrow_{\mathcal{R}}$ is confluent iff $\Rightarrow_{\mathcal{R}}$ is confluent

Proof: (a) We write $s \rightarrow_{\mathcal{R}}^{\pi} t$ if π is the position of the rewrite step.

Then: $s \rightarrow_{\mathcal{R}}^{\pi} t \rightsquigarrow s \Rightarrow_{\mathcal{R}} t$ using $\Pi = \{\pi\}$.

Now let $\Pi = \{\pi_1, \dots, \pi_n\}$.

Then: $s \Rightarrow_{\mathcal{R}} t \rightsquigarrow s \xrightarrow{\pi_1}_{\mathcal{R}} s_1 \xrightarrow{\pi_2}_{\mathcal{R}} s_2 \xrightarrow{\pi_3}_{\mathcal{R}} \dots \xrightarrow{\pi_n}_{\mathcal{R}} t$
 $\rightsquigarrow s \rightarrow_{\mathcal{R}}^* t$

(b) $\rightarrow_{\mathcal{R}} \subseteq \Rightarrow_{\mathcal{R}} \subseteq \rightarrow_{\mathcal{R}}^*$ (by (a))

This implies $\rightarrow_{\mathcal{R}}^{\#} \subseteq \rightarrow_{\mathcal{R}}^{\#} \subseteq \underbrace{(\rightarrow_{\mathcal{R}}^{\#})^{\#}}_{\rightarrow_{\mathcal{R}}^{\#}}$

Thus: $\rightarrow_{\mathcal{R}}^{\#} = \rightarrow_{\mathcal{R}}^{\#}$.

Hence: $\rightarrow_{\mathcal{R}}$ is confluent

iff $\rightarrow_{\mathcal{R}}^{\#}$ is confluent

iff $\Rightarrow_{\mathcal{R}}^{\#}$ is confluent

iff $\Rightarrow_{\mathcal{R}}$ is confluent

□

Thm 5.3.10 (Orthogonality \Rightarrow Confluence)

Every orthogonal TRS is confluent.

Proof: One can show that for every orthogonal TRS \mathcal{R} , $\Rightarrow_{\mathcal{R}}$ is strongly confluent.

Then: $\Rightarrow_{\mathcal{R}}$ is strongly confluent

$\leadsto \Rightarrow_{\mathcal{R}}$ is confluent (by Thm. 5.3.6)

$\leadsto \rightarrow_{\mathcal{R}}$ is confluent (Lemma 5.3.9 (b))

□