5.3 Confluence Without Termination

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In prog. languages: evaluation strategy may influence efficiency, but it should not influence the result.

But programs do not always terminate.

$\Rightarrow$ Programs should impose syntactic restrictions that ensure confluence even in case of non-termination.

(Indeed, languages like Haskell impose such restrictions.)

Requiring that all critical pairs are joinable is sufficient for local confluence, but not for confluence if the TRS is not terminating.

First Idea: Require that there must not be any critical pairs.

Def 5.3.1. (Non-Overlapping TRSs)

A TRS is non-overlapping iff it has no critical pairs.

Unfortunately, this is not enough to ensure confluence.

Ex. 5.3.2

$$eq(x, x) \to true$$

$$eq(x, succ(x)) \to false$$

$$a \to succ(a)$$

This TRS is non-overlapping, but not confluent:

$$\begin{tikzcd}[column sep=small] & eq(a,\underline{a}) \arrow[dl] \arrow[dr] & & \\ true & & eq(a, succ(a)) \arrow[dr] & \\ & & & false \end{tikzcd}$$

The problem is that the left-hand sides of the first 2 rules are not <u>linear</u>.

<u>Def 533</u> (Left-linearity, Orthogonality)
A term is <u>linear</u> iff it does not contain multiple occurrences of the same variable. A TRS is <u>left-linear</u> iff all left-hand sides of its rules are linear. A TRS is orthogonal iff it is non-overlapping and left-linear.

We will show that: orthogonal $\leadsto$ confluent
(even for non-terminating TRSs).
This is the reason why left-linearity is required in functional prog. languages.
To show this: introduce "strong confluence".

<u>Def 534</u> (Strong Confluence)
A relation $\to$ on a set M is <u>strongly confluent</u> iff for all $p, s, t \in M$:

if $p \to s$ and $p \to t$,

then there exists a $q \in M$ such that

$s \to^{=} q$ and $t \to^{=} q$.

Here, "$\to^{=}$" is the reflexive closure of "$\to$",
i.e., $s \to^{=} q$ iff $s \to q$ or $s = q$.

A TRS $\mathcal{R}$ is strongly confluent iff $\to_{\mathcal{R}}$ is strongly confluent.

Confluence
$$\begin{tikzcd} & p \arrow[dl, "*"'] \arrow[dr, "*"] & \\ s \arrow[dr, "*"'] & & t \arrow[dl, "*"] \\ & q & \end{tikzcd}$$

Local Confluence
$$\begin{tikzcd} & p \arrow[dl] \arrow[dr] & \\ s \arrow[dr, "*"'] & & t \arrow[dl, "*"] \\ & q & \end{tikzcd}$$

Strong Confl.
$$\begin{tikzcd} & p \arrow[dl] \arrow[dr] & \\ s \arrow[dr, "="'] & & t \arrow[dl, "="] \\ & q & \end{tikzcd}$$

Ex 5.3.5 $\mathcal{R} = \{ b \to a, b \to c, a \to b, c \to b \}$

$a \rightleftarrows b \rightleftarrows c$ $\qquad$ $\mathcal{R}$ is strongly confluent.

Thm 5.3.6 (Strong Confl. $\supset$ Confl.)
Let $\to$ be a strongly confluent relation. Then $\to$ is confluent.

Is $\to_{\mathcal{R}}$ strongly confluent for every orthogonal TRS $\mathcal{R}$? If yes, then this would prove that every orthogonal TRS is confluent.

Ex: $\mathcal{R} = \{ f(x) \to g(x,x), a \to b \}$

$\mathcal{R}$ is orthogonal. (and confluent, since it is terminating and has no critical pairs).

$$\begin{tikzcd}
 & f(a) \arrow[dl] \arrow[dr] & \\
g(a,a) \arrow[d] & & f(b) \arrow[dl] \\
g(b,a) \arrow[r] & g(b,b) &
\end{tikzcd}$$

Not strongly confluent!

Idea: introduce a variant of the rewrite relation $\to_{\mathcal{R}}$ which allows to reduce subterms on pairwise independent positions <u>in parallel</u>. This variant is indeed strongly confluent for orthogonal TRSs $\mathcal{R}$.

<u>Def 5.37</u> (Parallel Rewrite Relation)

For a TRS $\mathcal{R}$, we define the parallel rewrite relation $\rightrightarrows_{\mathcal{R}}$ as follows: Let $s \rightrightarrows_{\mathcal{R}} t$ hold iff there is a set $\Pi = \{\pi_1, \ldots, \pi_n\} \subseteq Occ(s)$ with $n \neq 0$ and for each π_i there is a rule $l_i \to r_i \in \mathcal{R}$ and a substitution σ_i such that

$$s|_{\pi_i} = l_i\sigma_i \text{ and } t = s[r_1\sigma_1]_{\pi_1} \ldots [r_n\sigma_n]_{\pi_n}.$$

Here, $\pi_i \perp \pi_j$ must hold for all $1 \leq i < j \leq n$.
(All elements of Π must be pairwise independent.)

<u>Ex. 5.38.</u> $\mathcal{R} = \{f(x) \to g(x,x),\ a \to b\}$

$g(a,a)$ has the positions $\Pi = \{1, 2\}$.

Thus $g(a,a) \Rrightarrow_{\mathcal{R}} g(b,b)$.

To prove that orthogonality implies confluence, we show:

$\mathcal{R}$ orthogonal $\overset{\text{TODO}}{\curvearrowright}$ $\Rrightarrow_{\mathcal{R}}$ is strongly confluent

$\underset{\text{Thm 5.3.6}}{\curvearrowright}$ $\Rrightarrow_{\mathcal{R}}$ is confluent

$\overset{\text{TODO}}{\curvearrowright}$ $\to_{\mathcal{R}}$ is confluent

Lemma 5.3.9 (Confluence of $\to_{\mathcal{R}}$ and $\Rrightarrow_{\mathcal{R}}$)

Let $\mathcal{R}$ be a TRS.

(a) $\to_{\mathcal{R}} \;\subseteq\; \Rrightarrow_{\mathcal{R}} \;\subseteq\; \to_{\mathcal{R}}^{*}$

(b) $\to_{\mathcal{R}}$ is confluent iff $\Rrightarrow_{\mathcal{R}}$ is confluent

Proof: (a) We write $s \to_{\mathcal{R}}^{\pi} t$ if π is the position of the rewrite step.

Then: $s \to_{\mathcal{R}}^{\pi} t \;\curvearrowright\; s \Rrightarrow_{\mathcal{R}} t$ using $\Pi = \{\pi\}$.

Now let $\Pi = \{\pi_1, \dots, \pi_n\}$.

Then: $s \Rrightarrow_{\mathcal{R}} t \;\curvearrowright\; s \to_{\mathcal{R}}^{\pi_1} s_1 \to_{\mathcal{R}}^{\pi_2} s_2 \to_{\mathcal{R}}^{\pi_3} \dots \to_{\mathcal{R}}^{\pi_n} t$

$\curvearrowright\; s \to_{\mathcal{R}}^{*} t$

(b) $\to_{\mathcal{R}} \;\subseteq\; \Rrightarrow_{\mathcal{R}} \;\subseteq\; \to_{\mathcal{R}}^{*}$ (by (a))

This implies $\to_{\mathcal{R}}^{*} \subseteq \rightrightarrows_{\mathcal{R}}^{*} \subseteq \underbrace{\left(\to_{\mathcal{R}}^{*}\right)^{*}}_{\to_{\mathcal{R}}^{*}}$

Thus: $\to_{\mathcal{R}}^{*} = \rightrightarrows_{\mathcal{R}}^{*}$.

Hence: $\to_{\mathcal{R}}$ is confluent

iff $\to_{\mathcal{R}}^{*}$ is confluent

iff $\rightrightarrows_{\mathcal{R}}^{*}$ is confluent

iff $\rightrightarrows_{\mathcal{R}}$ is confluent

$\square$

Thm 5.3.10 (Orthogonality $\leadsto$ Confluence)

Every orthogonal TRS is confluent.

Proof: One can show that for every orthogonal TRS $\mathcal{R}$, $\rightrightarrows_{\mathcal{R}}$ is strongly confluent.

Then: $\rightrightarrows_{\mathcal{R}}$ is strongly confluent

$\leadsto$ $\rightrightarrows_{\mathcal{R}}$ is confluent (by Thm. 5.3.6)

$\leftrightsquigarrow$ $\to_{\mathcal{R}}$ is confluent (lemma 5.3.9 (b)).

$\square$